

## CHAPTER

# 9

# The Pythagorean Theorem



*But serving up an action, suggesting the dynamic in the static, has become a hobby of mine . . . The “flowing” on that motionless plane holds my attention to such a degree that my preference is to try and make it into a cycle.*

M. C. ESCHER

*Waterfall*, M. C. Escher, 1961  
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## OBJECTIVES

In this chapter you will

- discover the Pythagorean Theorem, one of the most important concepts in mathematics
- use the Pythagorean Theorem to calculate the distance between any two points
- use conjectures related to the Pythagorean Theorem to solve problems

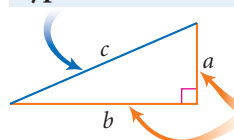
*I am not young enough to know everything.*

OSCAR WILDE

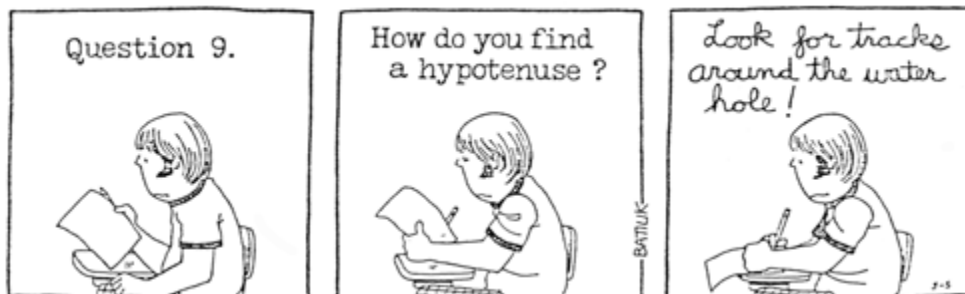
# The Theorem of Pythagoras

In a right triangle, the side opposite the right angle is called the **hypotenuse**. The other two sides are called **legs**. In the figure at right,  $a$  and  $b$  represent the lengths of the legs, and  $c$  represents the length of the hypotenuse.

In a right triangle, the side opposite the right angle is called the **hypotenuse**, here with length  $c$ .



The other two sides are **legs**, here with lengths  $a$  and  $b$ .



FUNKY WINKERBEAN by Batiuk. Reprinted with special permission of North America Syndicate.

There is a special relationship between the lengths of the legs and the length of the hypotenuse. This relationship is known today as the **Pythagorean Theorem**.



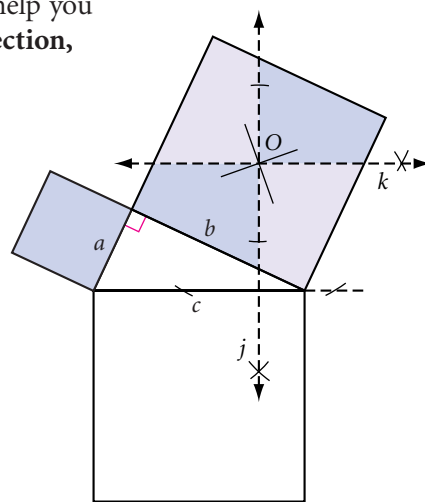
## Investigation

### The Three Sides of a Right Triangle

#### You will need

- scissors
- a compass
- a straightedge
- patty paper

- Step 1** Construct a scalene right triangle in the middle of your paper. Label the hypotenuse  $c$  and the legs  $a$  and  $b$ . Construct a square on each side of the triangle.
- Step 2** To locate the center of the square on the longer leg, draw its diagonals. Label the center  $O$ .
- Step 3** Through point  $O$ , construct line  $j$  perpendicular to the hypotenuse and line  $k$  perpendicular to line  $j$ . Line  $k$  is parallel to the hypotenuse. Lines  $j$  and  $k$  divide the square on the longer leg into four parts.
- Step 4** Cut out the square on the shorter leg and the four parts of the square on the longer leg. Arrange them to exactly cover the square on the hypotenuse.

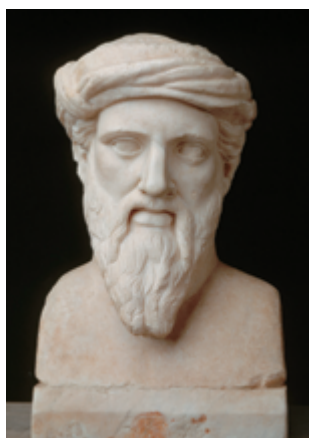


## The Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse. If  $a$  and  $b$  are the lengths of the legs, and  $c$  is the length of the hypotenuse, then  $\underline{\quad ? \quad}$ .

### History

#### CONNECTION



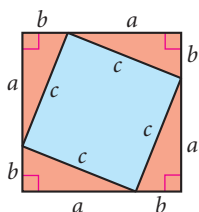
Pythagoras of Samos (ca. 569–475 B.C.E.), depicted in this statue, is often described as “the first pure mathematician.” Samos was a principal commercial center of Greece and is located on the island of Samos in the Aegean Sea. The ancient town of Samos now lies in ruins, as shown in the photo at right.

Mysteriously, none of Pythagoras’s writings still exist, and we know very little about his life. He founded a mathematical society in Croton, in what is now Italy, whose members discovered irrational numbers and the five regular solids. They proved what is now called the Pythagorean Theorem, although it was discovered and used 1000 years earlier by the Chinese and Babylonians. Some math historians believe that the ancient Egyptians also used a special case of this property to construct right angles.



A **theorem** is a conjecture that has been proved. Demonstrations like the one in the investigation are the first step toward proving the Pythagorean Theorem.

Believe it or not, there are more than 200 proofs of the Pythagorean Theorem. Elisha Scott Loomis’s *Pythagorean Proposition*, first published in 1927, contains original proofs by Pythagoras, Euclid, and even Leonardo da Vinci and U. S. President James Garfield. One well-known proof of the Pythagorean Theorem is included below. You will complete another proof as an exercise.



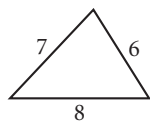
### Paragraph Proof: The Pythagorean Theorem

You need to show that  $a^2 + b^2$  equals  $c^2$  for the right triangles in the figure at left. The area of the entire square is  $(a + b)^2$  or  $a^2 + 2ab + b^2$ . The area of any triangle is  $\left(\frac{1}{2}\right)ab$ , so the sum of the areas of the four triangles is  $2ab$ . The area of the quadrilateral in the center is  $(a^2 + 2ab + b^2) - 2ab$ , or  $a^2 + b^2$ .

If the quadrilateral in the center is a square then its area also equals  $c^2$ . You now need to show that it is a square. You know that all the sides have length  $c$ , but you also need to show that the angles are right angles. The two acute angles in the right triangle, along with any angle of the quadrilateral, add up to  $180^\circ$ . The acute angles in a right triangle add up to  $90^\circ$ . Therefore the quadrilateral angle measures  $90^\circ$  and the quadrilateral is a square. If it is a square with side length  $c$ , then its area is  $c^2$ . So,  $a^2 + b^2 = c^2$ , which proves the Pythagorean Theorem. ■

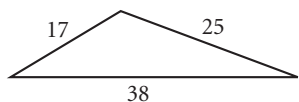
The Pythagorean Theorem works for right triangles, but does it work for all triangles? A quick check demonstrates that it doesn't hold for other triangles.

Acute triangle

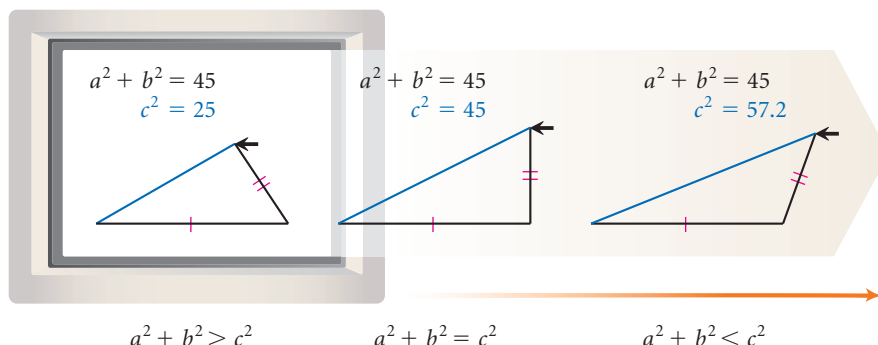


$$6^2 + 7^2 > 8^2$$

Obtuse triangle



$$17^2 + 25^2 < 38^2$$



For an interactive version of this sketch, visit [www.keymath.com/DG](http://www.keymath.com/DG).

Let's look at a few examples to see how you can use the Pythagorean Theorem to find the distance between two points.

### EXAMPLE A

How high up on the wall will a 20-foot ladder touch if the foot of the ladder is placed 5 feet from the wall?

#### ► Solution

The ladder is the hypotenuse of a right triangle, so  $a^2 + b^2 = c^2$ .

$$(5)^2 + (h)^2 = (20)^2$$

Substitute.

$$25 + h^2 = 400$$

Multiply.

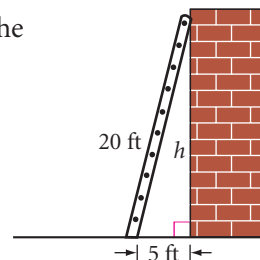
$$h^2 = 375$$

Subtract 25 from both sides.

$$h = \sqrt{375} \approx 19.4$$

Take the square root of each side.

The top of the ladder will touch the wall about 19.4 feet up from the ground.



Notice that the exact answer in Example A is  $\sqrt{375}$ . However, this is a practical application, so you need to calculate the approximate answer.

### EXAMPLE B

Find the area of the rectangular rug if the width is 12 feet and the diagonal measures 20 feet.

#### ► Solution

Use the Pythagorean Theorem to find the length.

$$a^2 + b^2 = c^2$$

$$(12)^2 + (L)^2 = (20)^2$$

$$144 + L^2 = 400$$

$$L^2 = 256$$

$$L = \sqrt{256}$$

$$L = 16$$

The length is 16 feet. The area of the rectangle is  $12 \cdot 16$ , or 192 square feet.

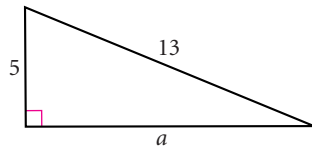




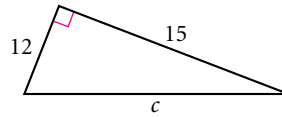
# EXERCISES

In Exercises 1–11, find each missing length. All measurements are in centimeters. Give approximate answers accurate to the nearest tenth of a centimeter.

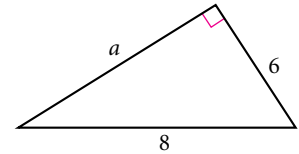
1.  $a = ?$



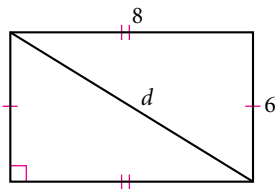
2.  $c \approx ?$



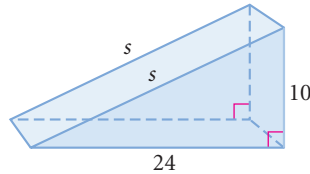
3.  $a \approx ?$



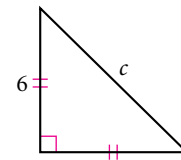
4.  $d = ?$



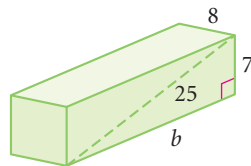
5.  $s = ?$



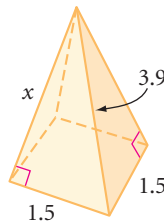
6.  $c \approx ?$  (h)



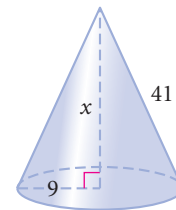
7.  $b = ?$



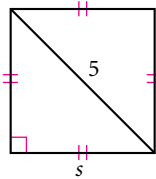
8.  $x = ?$



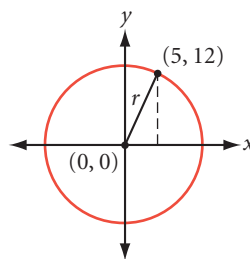
9. The base is a circle.  
 $x = ?$



10.  $s \approx ?$



11.  $r = ?$  (h)



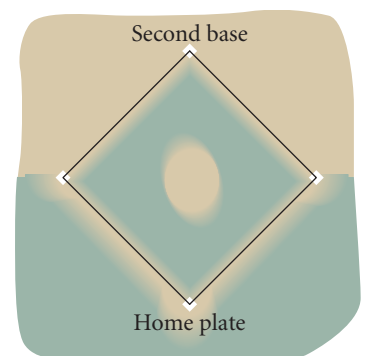
12. A baseball infield is a square, each side measuring 90 feet. To the nearest foot, what is the distance from home plate to second base?

13. The diagonal of a square measures 32 meters. What is the area of the square? (h)

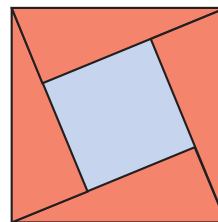
14. What is the length of the diagonal of a square whose area is  $64 \text{ cm}^2$ ?

15. The lengths of the three sides of a right triangle are consecutive integers. Find them. (h)

16. A rectangular garden 6 meters wide has a diagonal measuring 10 meters. Find the perimeter of the garden.



17. One very famous proof of the Pythagorean Theorem is by the Hindu mathematician Bhaskara. It is often called the “Behold” proof because, as the story goes, Bhaskara drew the diagram at right and offered no verbal argument other than to exclaim, “Behold.” Use algebra to fill in the steps, explaining why this diagram proves the Pythagorean Theorem. [h](#)

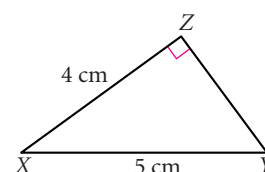
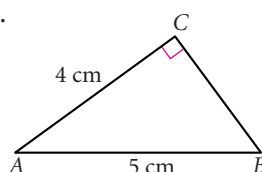


## History

### CONNECTION

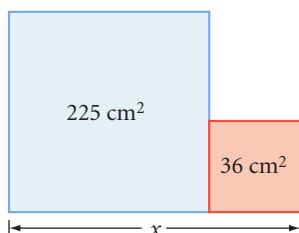
Bhaskara (1114–1185, India) was one of the first mathematicians to gain a thorough understanding of number systems and how to solve equations, several centuries before European mathematicians. He wrote six books on mathematics and astronomy, and led the astronomical observatory at Ujjain.

18. Is  $\triangle ABC \cong \triangle XYZ$ ? Explain your reasoning.

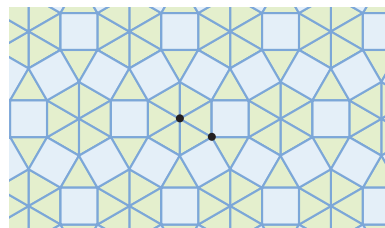


## Review

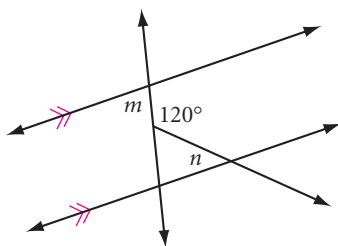
19. The two quadrilaterals are squares. Find  $x$ .



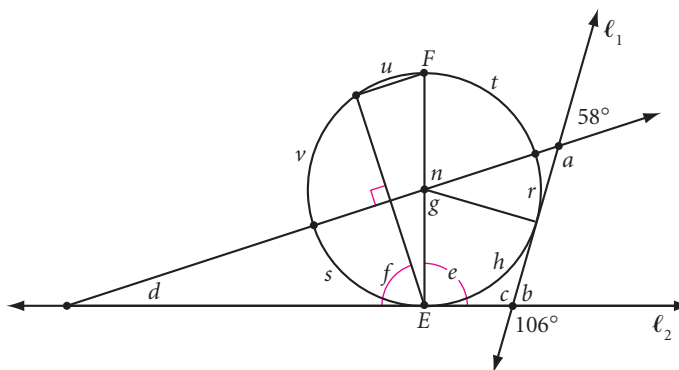
20. Give the vertex arrangement for the 2-uniform tessellation.



21. Explain why  $m + n = 120^\circ$ .



22. Calculate each lettered angle, measure, or arc.  $\overline{EF}$  is a diameter;  $\ell_1$  and  $\ell_2$  are tangents.



# project

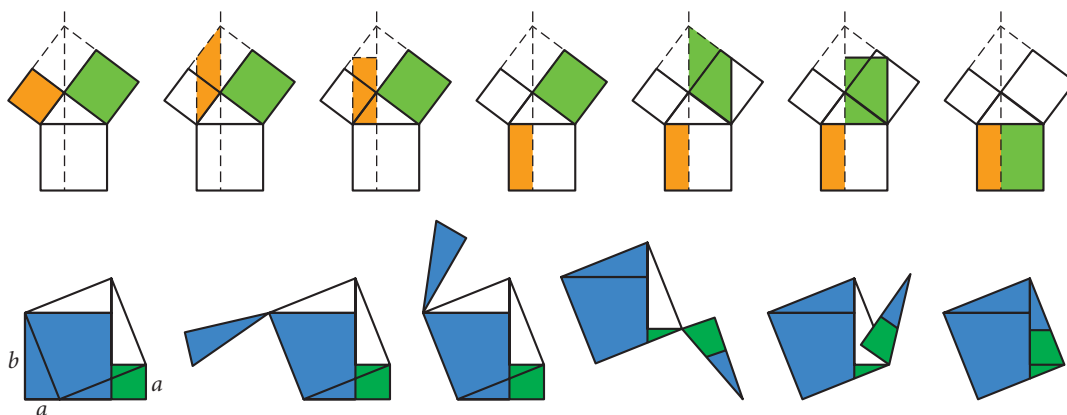
## CREATING A GEOMETRY FLIP BOOK

Have you ever fanned the pages of a flip book and watched the pictures seem to move? Each page shows a picture slightly different from the previous one. Flip books are basic to animation technique. For more information about flip books, see [www.keymath.com/DG](http://www.keymath.com/DG).

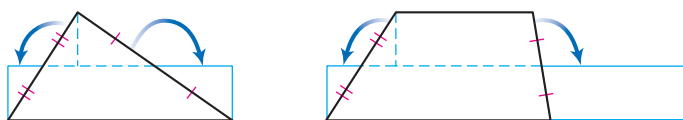


These five frames start off the photo series titled *The Horse in Motion*, by photographer, innovator, and motion picture pioneer Eadweard Muybridge (1830–1904).

Here are two dissections that you can animate to demonstrate the Pythagorean Theorem. (You used another dissection in the Investigation The Three Sides of a Right Triangle.)



You could also animate these drawings to demonstrate area formulas.



Choose one of the animations mentioned above and create a flip book that demonstrates it. Be ready to explain how your flip book demonstrates the formula you chose.

Here are some practical tips.

- ▶ Draw your figures in the same position on each page so they don't jump around when the pages are flipped. Use graph paper or tracing paper to help.
- ▶ The smaller the change from picture to picture, and the more pictures there are, the smoother the motion will be.
- ▶ Label each picture so that it's clear how the process works.

# The Converse of the Pythagorean Theorem

Any time you see someone more successful than you are, they are doing something you aren't.

MALCOLM X

In Lesson 9.1, you saw that if a triangle is a right triangle, then the square of the length of its hypotenuse is equal to the sum of the squares of the lengths of the two legs. What about the converse? If  $x$ ,  $y$ , and  $z$  are the lengths of the three sides of a triangle and they satisfy the Pythagorean equation,  $a^2 + b^2 = c^2$ , must the triangle be a right triangle? Let's find out.



## Investigation Is the Converse True?

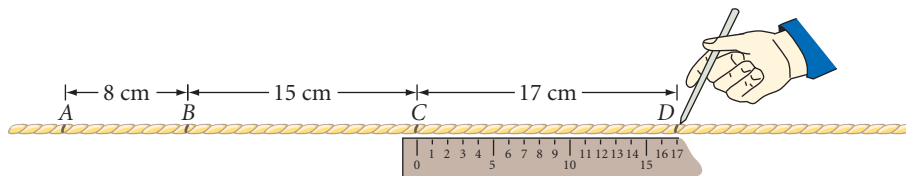
### You will need

- string
- a ruler
- paper clips
- a piece of patty paper

Three positive integers that work in the Pythagorean equation are called **Pythagorean triples**. For example, 8-15-17 is a Pythagorean triple because  $8^2 + 15^2 = 17^2$ . Here are nine sets of Pythagorean triples.

3-4-5	5-12-13	7-24-25	8-15-17
6-8-10	10-24-26		16-30-34
9-12-15			
12-16-20			

- Step 1** Select one set of Pythagorean triples from the list above. Mark off four points,  $A$ ,  $B$ ,  $C$ , and  $D$ , on a string to create three consecutive lengths from your set of triples.



- Step 2** Loop three paper clips onto the string. Tie the ends together so that points  $A$  and  $D$  meet.
- Step 3** Three group members should each pull a paper clip at point  $A$ ,  $B$ , or  $C$  to stretch the string tight.



- Step 4 | With your paper, check the largest angle. What type of triangle is formed?
- Step 5 | Select another set of triples from the list. Repeat Steps 1–4 with your new lengths.
- Step 6 | Compare results in your group. State your results as your next conjecture.

### Converse of the Pythagorean Theorem

C-83

If the lengths of the three sides of a triangle satisfy the Pythagorean equation, then the triangle ?.

This ancient Babylonian tablet, called Plimpton 322, dates sometime between 1900 and 1600 B.C.E. It suggests several advanced Pythagorean triples, such as 1679-2400-2929.



#### History

#### CONNECTION

Some historians believe Egyptian “rope stretchers” used the Converse of the Pythagorean Theorem to help reestablish land boundaries after the yearly flooding of the Nile and to help construct the pyramids. Some ancient tombs show workers carrying ropes tied with equally spaced knots. For example, 13 equally spaced knots would divide the rope into 12 equal lengths. If one person held knots 1 and 13 together, and two others held the rope at knots 4 and 8 and stretched it tight, they could have created a 3-4-5 right triangle.



The proof of the Converse of the Pythagorean Theorem is very interesting because it is one of the few instances where the original theorem is used to prove the converse. Let's take a look. One proof is started for you below. You will finish it as an exercise.

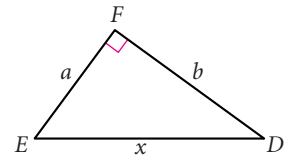
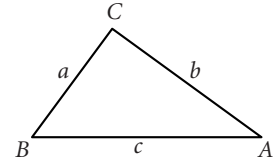
### Proof: Converse of the Pythagorean Theorem

**Conjecture:** If the lengths of the three sides of a triangle work in the Pythagorean equation, then the triangle is a right triangle.

**Given:**  $a, b, c$  are the lengths of the sides of  $\triangle ABC$  and  $a^2 + b^2 = c^2$

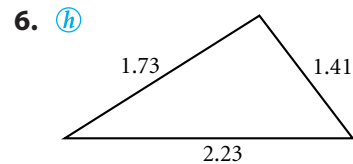
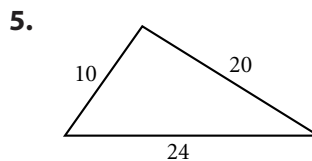
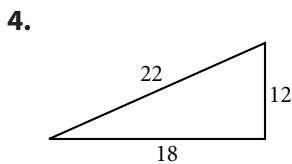
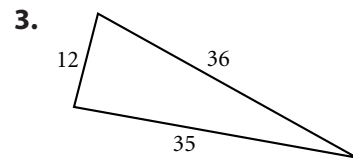
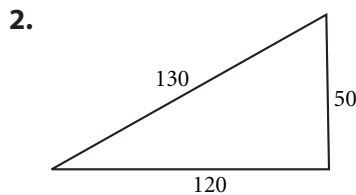
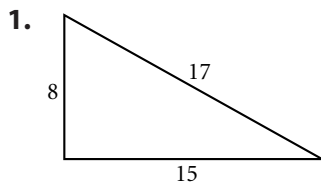
**Show:**  $\triangle ABC$  is a right triangle

**Plan:** Begin by constructing a second triangle, right triangle  $DEF$  (with  $\angle F$  a right angle), with legs of lengths  $a$  and  $b$  and hypotenuse of length  $x$ . The plan is to show that  $x = c$ , so that the triangles are congruent. Then show that  $\angle C$  and  $\angle F$  are congruent. Once you show that  $\angle C$  is a right angle, then  $\triangle ABC$  is a right triangle and the proof is complete.



## EXERCISES

In Exercises 1–6, use the Converse of the Pythagorean Theorem to determine whether each triangle is a right triangle.



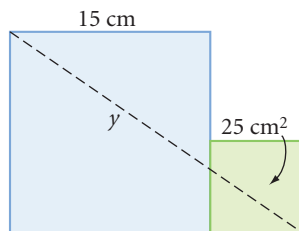
In Exercises 7 and 8, use the Converse of the Pythagorean Theorem to solve each problem.

7. Is a triangle with sides measuring 9 feet, 12 feet, and 18 feet a right triangle?
8. A window frame that seems rectangular has height 408 cm, length 306 cm, and one diagonal with length 525 cm. Is the window frame really rectangular? Explain.

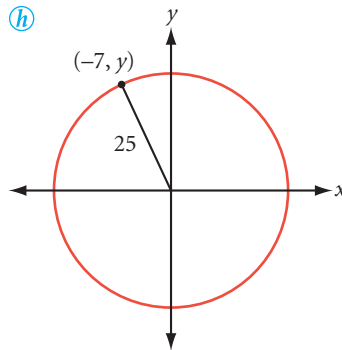


In Exercises 9–11, find  $y$ .

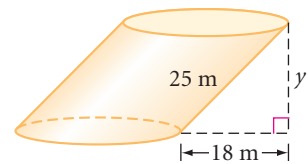
9. Both quadrilaterals are squares.



10.  $\textcircled{h}$



- 11.



12. The lengths of the three sides of a right triangle are consecutive even integers. Find them.  $\textcircled{h}$

13. Find the area of a right triangle with hypotenuse length 17 cm and one leg length 15 cm.

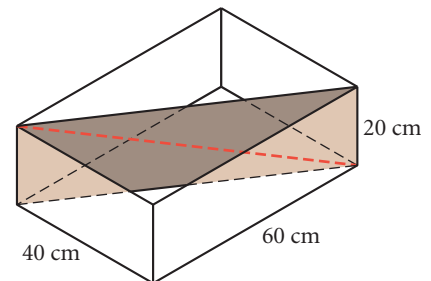
14. How high on a building will a 15-foot ladder touch if the foot of the ladder is 5 feet from the building?

15. The congruent sides of an isosceles triangle measure 6 cm, and the base measures 8 cm. Find the area.

16. Find the amount of fencing in linear feet needed for the perimeter of a rectangular lot with a diagonal length 39 m and a side length 36 m.

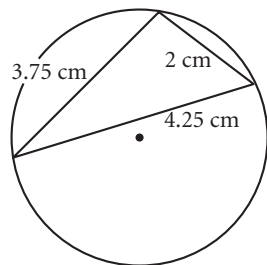
17. A rectangular piece of cardboard fits snugly on a diagonal in this box.

- What is the area of the cardboard rectangle?
- What is the length of the diagonal of the cardboard rectangle?

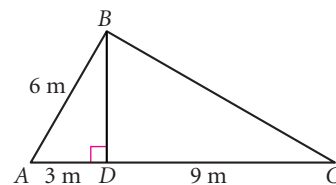


18. Look back at the start of the proof of the Converse of the Pythagorean Theorem. Copy the conjecture, the given, the show, the plan, and the two diagrams. Use the plan to complete the proof.

19. What's wrong with this picture?

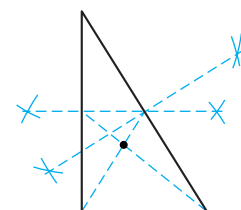


20. Explain why  $\triangle ABC$  is a right triangle.

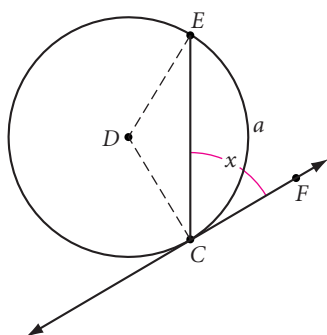


## Review

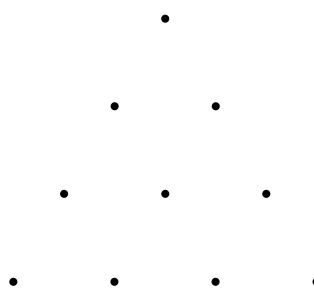
21. Identify the point of concurrency from the construction marks.



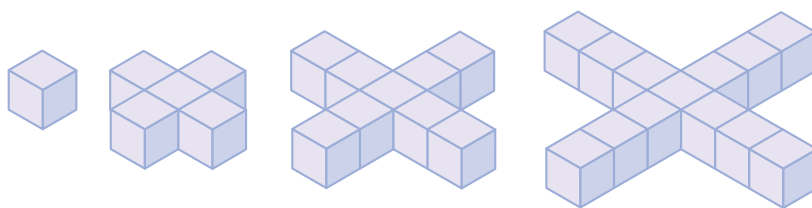
22. Line  $CF$  is tangent to circle  $D$  at  $C$ . The arc measure of  $\widehat{CE}$  is  $a$ . Explain why  $x = \left(\frac{1}{2}\right)a$ . (h)



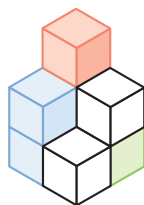
23. What is the probability of randomly selecting three points that form an isosceles triangle from the 10 points in this isometric grid?



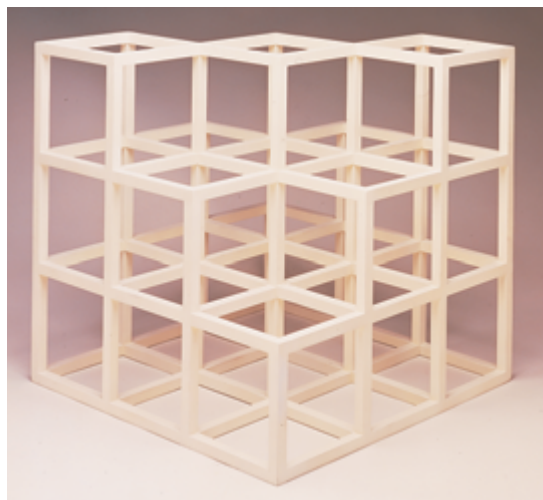
24. If the pattern of blocks continues, what will be the surface area of the 50th solid in the pattern?



25. Sketch the solid shown, but with the two blue cubes removed and the red cube moved to cover the visible face of the green cube.



The outlines of stacked cubes create a visual impact in this untitled module unit sculpture by conceptual artist Sol Lewitt.



## IMPROVING YOUR ALGEBRA SKILLS

### Algebraic Sequences III

Find the next three terms of this algebraic sequence.

$$x^9, 9x^8y, 36x^7y^2, 84x^6y^3, 126x^5y^4, 126x^4y^5, 84x^3y^6, \underline{\quad}, \underline{\quad}, \underline{\quad}$$





# Radical Expressions

When you work with the Pythagorean Theorem, you often get radical expressions, such as  $\sqrt{50}$ . Until now you may have left these expressions as radicals, or you may have found a decimal approximation using a calculator. Some radical expressions can be simplified. To simplify a square root means to take the square root of any perfect-square factors of the number under the radical sign. Let's look at an example.

**EXAMPLE A** | Simplify  $\sqrt{50}$ .

► **Solution**

One way to simplify a square root is to look for perfect-square factors.

The largest perfect-square factor of 50 is 25.

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$$

25 is a perfect square, so you can take its square root.

Another approach is to factor the number as far as possible with prime factors.

Write 50 as a set of prime factors.

Look for any square factors (factors that appear twice).

$$\sqrt{50} = \sqrt{5 \cdot 5 \cdot 2} = \sqrt{5^2 \cdot 2} = \sqrt{5^2} \cdot \sqrt{2} = 5\sqrt{2}$$

Squaring and taking the square root are inverse operations—they undo each other. So,  $\sqrt{5^2}$  equals 5.

You might argue that  $5\sqrt{2}$  doesn't look any simpler than  $\sqrt{50}$ . However, in the days before calculators with square root buttons, mathematicians used paper-and-pencil algorithms to find approximate values of square roots. Working with the smallest possible number under the radical made the algorithms easier to use.

Giving an exact answer to a problem involving a square root is important in a number of situations. Some patterns are easier to discover with simplified square roots than with decimal approximations. Standardized tests often express answers in simplified form. And when you multiply radical expressions, you often have to simplify the answer.

**EXAMPLE B** | Multiply  $3\sqrt{6}$  by  $5\sqrt{2}$ .

**► Solution** | To multiply radical expressions, associate and multiply the quantities outside the radical sign, and associate and multiply the quantities inside the radical sign.  
 $(3\sqrt{6})(5\sqrt{2}) = 3 \cdot 5 \cdot \sqrt{6 \cdot 2} = 15 \cdot \sqrt{12} = 15 \cdot \sqrt{4 \cdot 3} = 15 \cdot 2\sqrt{3} = 30\sqrt{3}$

EXERCISES

In Exercises 1–5, express each product in its simplest form.

1.  $(\sqrt{3})(\sqrt{2})$
2.  $(\sqrt{5})^2$
3.  $(3\sqrt{6})(2\sqrt{3})$
4.  $(7\sqrt{3})^2$
5.  $(2\sqrt{2})^2$

In Exercises 6–20, express each square root in its simplest form.

6.  $\sqrt{18}$
7.  $\sqrt{40}$
8.  $\sqrt{75}$
9.  $\sqrt{85}$
10.  $\sqrt{96}$
11.  $\sqrt{576}$
12.  $\sqrt{720}$
13.  $\sqrt{722}$
14.  $\sqrt{784}$
15.  $\sqrt{828}$
16.  $\sqrt{2952}$
17.  $\sqrt{5248}$
18.  $\sqrt{8200}$
19.  $\sqrt{11808}$
20.  $\sqrt{16072}$
21. What is the next term in the pattern?  $\sqrt{2952}, \sqrt{5248}, \sqrt{8200}, \sqrt{11808}, \sqrt{16072}, \dots$

IMPROVING YOUR VISUAL THINKING SKILLS

Folding Cubes II

Each cube has designs on three faces. When unfolded, which figure at right could it become?

1.

A.

B.

C.

D.

2.

A.

B.

C.

D.



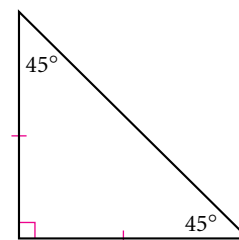
*In an isosceles triangle, the sum of the square roots of the two equal sides is equal to the square root of the third side.*

THE SCARECROW IN THE 1939 FILM *THE WIZARD OF OZ*

# Two Special Right Triangles

In this lesson you will use the Pythagorean Theorem to discover some relationships between the sides of two special right triangles.

One of these special triangles is an isosceles right triangle, also called a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle. Each isosceles right triangle is half a square, so they show up often in mathematics and engineering. In the next investigation, you will look for a shortcut for finding the length of an unknown side in a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle.

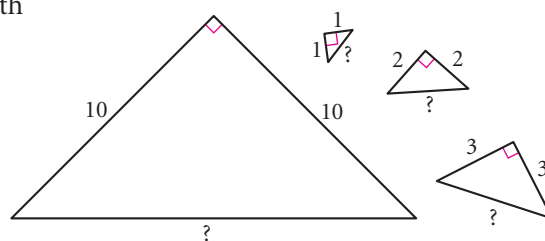
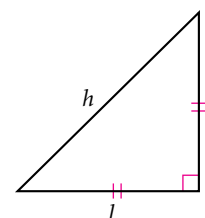


An isosceles right triangle



## Investigation 1 Isosceles Right Triangles

- Step 1 Sketch an isosceles right triangle. Label the legs  $l$  and the hypotenuse  $h$ .
- Step 2 Pick any integer for  $l$ , the length of the legs. Use the Pythagorean Theorem to find  $h$ . Simplify the square root.
- Step 3 Repeat Step 2 with several different values for  $l$ . Share results with your group. Do you see any pattern in the relationship between  $l$  and  $h$ ?
- Step 4 State your next conjecture in terms of length  $l$ .

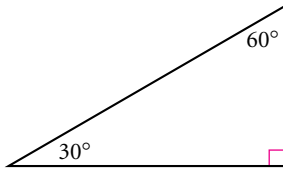
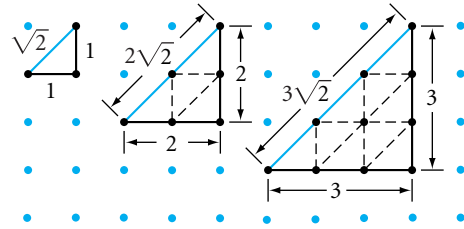


### Isosceles Right Triangle Conjecture

C-84

In an isosceles right triangle, if the legs have length  $l$ , then the hypotenuse has length  $\underline{\hspace{1cm}}$ .

You can also demonstrate this property on a geoboard or graph paper, as shown at right.



A 30°-60°-90° triangle

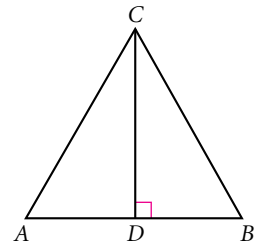
The other special right triangle is a 30°-60°-90° triangle. If you fold an equilateral triangle along one of its altitudes, the triangles you get are 30°-60°-90° triangles. A 30°-60°-90° triangle is half an equilateral triangle, so it also shows up often in mathematics and engineering. Let's see if there is a shortcut for finding the lengths of its sides.



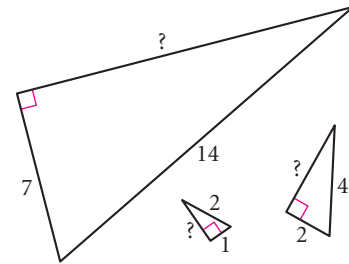
## Investigation 2

### 30°-60°-90° Triangles

Let's start by using a little deductive thinking to find the relationships in 30°-60°-90° triangles. Triangle  $ABC$  is equilateral, and  $\overline{CD}$  is an altitude.



- Step 1 What are  $m\angle A$  and  $m\angle B$ ? What are  $m\angle ACD$  and  $m\angle BCD$ ? What are  $m\angle ADC$  and  $m\angle BDC$ ?
- Step 2 Is  $\triangle ADC \cong \triangle BDC$ ? Why?
- Step 3 Is  $\overline{AD} \cong \overline{BD}$ ? Why? How do  $AC$  and  $AD$  compare?  
In a 30°-60°-90° triangle, will this relationship between the hypotenuse and the shorter leg always hold true? Explain.
- Step 4 Sketch a 30°-60°-90° triangle. Choose any integer for the length of the shorter leg. Use the relationship from Step 3 and the Pythagorean Theorem to find the length of the other leg. Simplify the square root.
- Step 5 Repeat Step 4 with several different values for the length of the shorter leg. Share results with your group. What is the relationship between the lengths of the two legs? You should notice a pattern in your answers.
- Step 6 State your next conjecture in terms of the length of the shorter leg,  $a$ .



### 30°-60°-90° Triangle Conjecture

C-85

In a 30°-60°-90° triangle, if the shorter leg has length  $a$ , then the longer leg has length  $\underline{\hspace{1cm}}$  and the hypotenuse has length  $\underline{\hspace{1cm}}$ .



You can use algebra to verify that the conjecture will hold true for any 30°-60°-90° triangle.

### Proof: 30°-60°-90° Triangle Conjecture

$$(2a)^2 = a^2 + b^2$$

$$4a^2 = a^2 + b^2$$

$$3a^2 = b^2$$

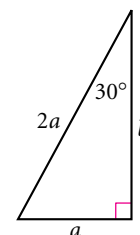
$$a\sqrt{3} = b$$

Start with the Pythagorean Theorem.

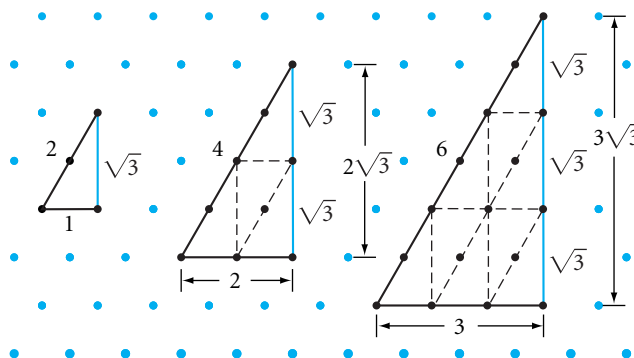
Square  $2a$ .

Subtract  $a^2$  from both sides.

Take the square root of both sides.



Although you investigated only integer values, the proof shows that any number, even a non-integer, can be used for  $a$ . You can also demonstrate this property for integer values on isometric dot paper.



## EXERCISES

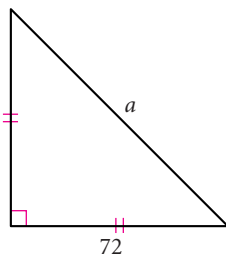
You will need



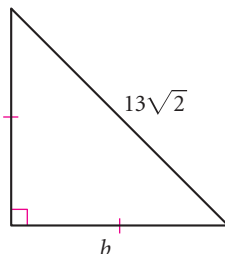
**Construction tools**  
for Exercises 19 and 20

In Exercises 1–8, use your new conjectures to find the unknown lengths. All measurements are in centimeters.

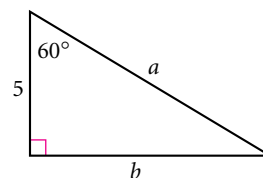
1.  $a = ?$



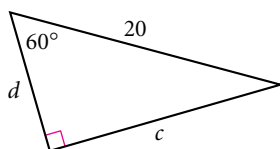
2.  $b = ?$  (h)



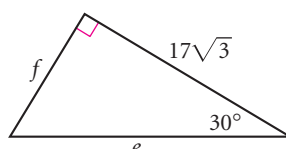
3.  $a = ?$ ,  $b = ?$



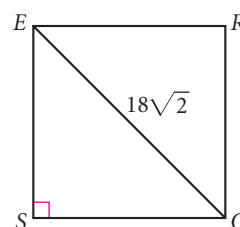
4.  $c = ?$ ,  $d = ?$  (h)



5.  $e = ?$ ,  $f = ?$

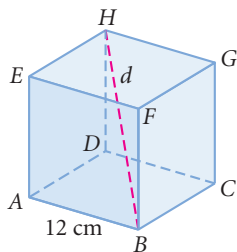


6. What is the perimeter of square SQRE?

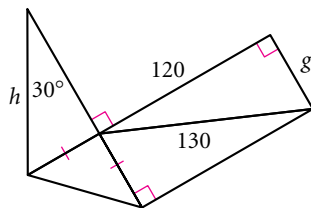


7. The solid is a cube.

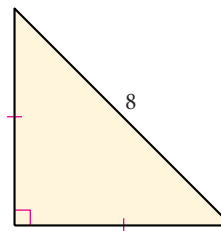
$d = ?$  (h)



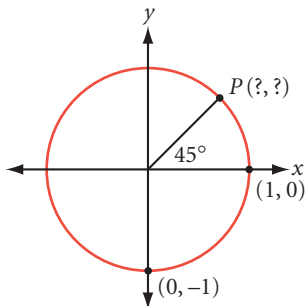
8.  $g = ?$ ,  $h = ?$



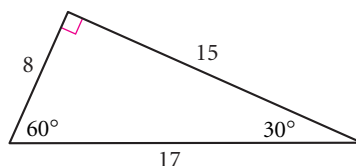
9. What is the area of the triangle? (h)



10. Find the coordinates of  $P$ .



11. What's wrong with this picture?

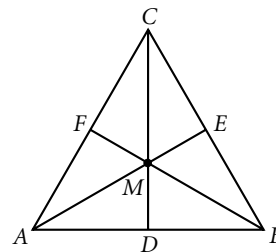


12. Sketch and label a figure to demonstrate that  $\sqrt{27}$  is equivalent to  $3\sqrt{3}$ . (Use isometric dot paper to aid your sketch.) (h)

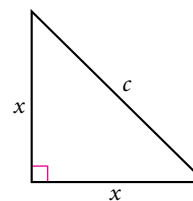
13. Sketch and label a figure to demonstrate that  $\sqrt{32}$  is equivalent to  $4\sqrt{2}$ . (Use square dot paper or graph paper.)

14. In equilateral triangle  $ABC$ ,  $\overline{AE}$ ,  $\overline{BF}$ , and  $\overline{CD}$  are all angle bisectors, medians, and altitudes simultaneously. These three segments divide the equilateral triangle into six overlapping  $30^\circ$ - $60^\circ$ - $90^\circ$  triangles and six smaller, non-overlapping  $30^\circ$ - $60^\circ$ - $90^\circ$  triangles.

- One of the overlapping triangles is  $\triangle CDB$ . Name the other five triangles that are congruent to it.
- One of the non-overlapping triangles is  $\triangle MDA$ . Name the other five triangles congruent to it.



15. Use algebra and deductive reasoning to show that the Isosceles Right Triangle Conjecture holds true for any isosceles right triangle. Use the figure at right.



16. Find the area of an equilateral triangle whose sides measure 26 meters. (h)

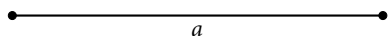
17. An equilateral triangle has an altitude that measures 26 meters. Find the area of the triangle to the nearest square meter.

18. Sketch the largest  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle that fits in a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle. What is the ratio of the area of the  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle to the area of the  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle?

## Review

**Construction** In Exercises 19 and 20, choose either patty paper or a compass and straightedge and perform the constructions.

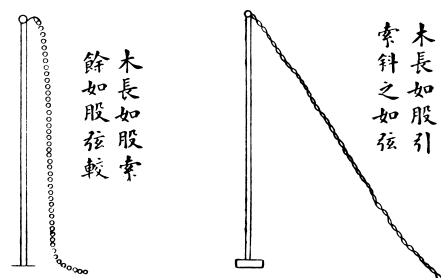
19. Given the segment with length  $a$  below, construct segments with lengths  $a\sqrt{2}$ ,  $a\sqrt{3}$ , and  $a\sqrt{5}$ . (h)



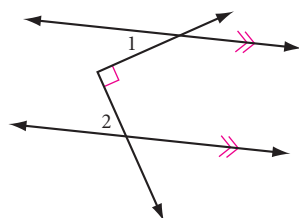
20. **Mini-Investigation** Draw a right triangle with sides of lengths 6 cm, 8 cm, and 10 cm. Locate the midpoint of each side. Construct a semicircle on each side with the midpoints of the sides as centers. Find the area of each semicircle. What relationship do you notice among the three areas?

21. The *Jiuzhang suanshu* is an ancient Chinese mathematics text of 246 problems. Some solutions use the *gou gu*, the Chinese name for what we call the Pythagorean Theorem. The *gou gu* reads  $(gou)^2 + (gu)^2 = (xian)^2$ . Here is a *gou gu* problem translated from the ninth chapter of *Jiuzhang*.

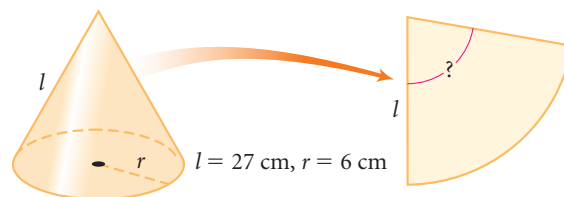
A rope hangs from the top of a pole with three *chih* of it lying on the ground. When it is tightly stretched so that its end just touches the ground, it is eight *chih* from the base of the pole. How long is the rope?



22. Explain why  $m\angle 1 + m\angle 2 = 90^\circ$ . (h)



23. The lateral surface area of the cone below is unwrapped into a sector. What is the angle at the vertex of the sector?



## IMPROVING YOUR VISUAL THINKING SKILLS

### Mudville Monsters

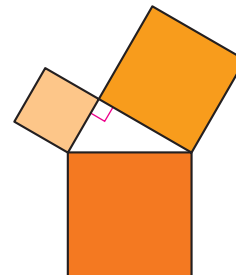
The 11 starting members of the Mudville Monsters football team and their coach, Osgood Gipper, have been invited to compete in the Smallville Punt, Pass, and Kick Competition. To get there, they must cross the deep Smallville River. The only way across is with a small boat owned by two very small Smallville football players. The boat holds just one Monster visitor or the two Smallville players. The Smallville players agree to help the Mudville players across if the visitors agree to pay \$5 each time the boat crosses the river. If the Monsters have a total of \$100 among them, do they have enough money to get all players and the coach to the other side of the river?



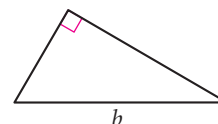
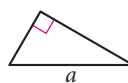
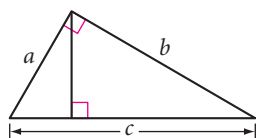
# Exploration

## A Pythagorean Fractal

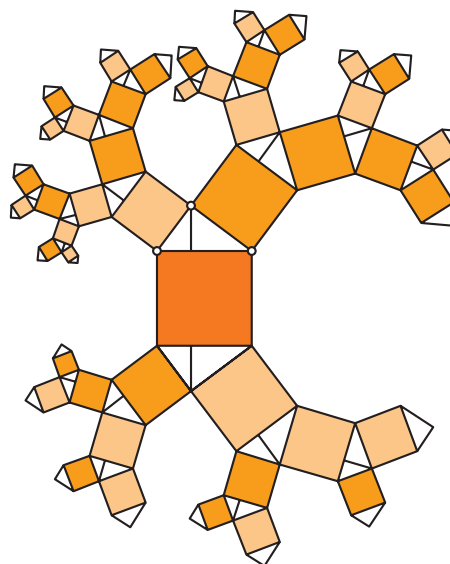
If you wanted to draw a picture to state the Pythagorean Theorem without words, you'd probably draw a right triangle with squares on each of the three sides. This is the way you first explored the Pythagorean Theorem in Lesson 9.1.



Another picture of the theorem is even simpler: a right triangle divided into two right triangles. Here, a right triangle with hypotenuse  $c$  is divided into two smaller triangles, the smaller with hypotenuse  $a$  and the larger with hypotenuse  $b$ . Clearly, their areas add up to the area of the whole triangle. What's surprising is that all three triangles have the same angle measures. Why? Though different in size, the three triangles all have the same shape. Figures that have the same shape but not necessarily the same size are called **similar figures**. You'll use these similar triangles to prove the Pythagorean Theorem in a later chapter.



A beautifully complex fractal combines both of these pictorial representations of the Pythagorean Theorem. The fractal starts with a right triangle with squares on each side. Then similar triangles are built onto the squares. Then squares are built onto the new triangles, and so on. In this exploration, you'll create this fractal.





## Activity

### The Right Triangle Fractal

#### You will need

- the worksheet  
The Right Triangle  
Fractal (optional)

The Geometer's Sketchpad software uses custom tools to save the steps of repeated constructions. They are very helpful for fractals like this one.

Step 1

Use The Geometer's Sketchpad to create the fractal on page 480. Follow the Procedure Note.

Notice that each square has two congruent triangles on two opposite sides. Use a reflection to guarantee that the triangles are congruent.

#### Procedure Note

1. Use the diameter of a circle and an inscribed angle to make a triangle that always remains a right triangle.
2. It is important to construct the altitude to the hypotenuse in each triangle in order to divide it into similar triangles.
3. Create custom tools to make squares and similar triangles repeatedly.

After you successfully make the Pythagorean fractal, you're ready to investigate its fascinating patterns.

Step 2

First, try dragging a vertex of the original triangle.

Step 3

Does the Pythagorean Theorem still apply to the branches of this figure? That is, does the sum of the areas of the branches on the legs equal the area of the branch on the hypotenuse? See if you can answer without actually measuring all the areas.

Step 4

Consider your original sketch to be a single right triangle with a square built on each side. Call this sketch Stage 0 of your fractal. Explore these questions.

- a. At Stage 1, you add three triangles and six squares to your construction. On a piece of paper, draw a rough sketch of Stage 1. How much area do you add to this fractal between Stage 0 and Stage 1? (Don't measure any areas to answer this.)
- b. Draw a rough sketch of Stage 2. How much area do you add between Stage 1 and Stage 2?
- c. How much area is added at any new stage?
- d. A true fractal exists only after an infinite number of stages. If you could build a true fractal based on the construction in this activity, what would be its total area?

Step 5

Give the same color and shade to sets of squares that are congruent. What do you notice about these sets of squares other than their equal area? Describe any patterns you find in sets of congruent squares.

Step 6

Describe any other patterns you can find in the Pythagorean fractal.

# Story Problems

You have learned that drawing a diagram will help you to solve difficult problems. By now you know to look for many special relationships in your diagrams, such as congruent polygons, parallel lines, and right triangles.

*You may be disappointed if you fail, but you are doomed if you don't try.*

BEVERLY SILLS



FUNKY WINKERBEAN by Batiuk. Reprinted with special permission of North America Syndicate.

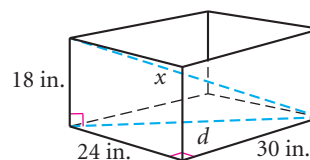
## EXAMPLE

What is the longest stick that will fit inside a 24-by-30-by-18-inch box?

## ► Solution

Draw a diagram.

You can lay a stick with length  $d$  diagonally at the bottom of the box. But you can position an even longer stick with length  $x$  along the diagonal of the box, as shown. How long is this stick?



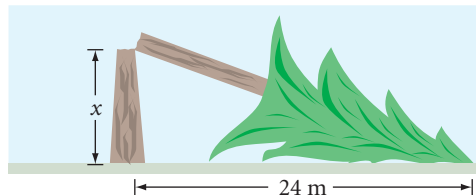
Both  $d$  and  $x$  are the hypotenuses of right triangles, but finding  $d^2$  will help you find  $x$ .

$$\begin{aligned} 30^2 + 24^2 &= d^2 & d^2 + 18^2 &= x^2 \\ 900 + 576 &= d^2 & 1476 + 18^2 &= x^2 \\ d^2 &= 1476 & 1476 + 324 &= x^2 \\ & & 1800 &= x^2 \\ & & x &\approx 42.4 \end{aligned}$$

The longest possible stick is about 42.4 in.

## EXERCISES

1. A giant California redwood tree 36 meters tall cracked in a violent storm and fell as if hinged. The tip of the once beautiful tree hit the ground 24 meters from the base. Researcher Red Woods wishes to investigate the crack. How many meters up from the base of the tree does he have to climb?  $\textcircled{h}$



2. Amir's sister is away at college, and he wants to mail her a 34 in. baseball bat. The packing service sells only one kind of box, which measures 24 in. by 2 in. by 18 in. Will the box be big enough?

3. Meteorologist Paul Windward and geologist Rhaina Stone are rushing to a paleontology conference in Pecos Gulch. Paul lifts off in his balloon at noon from Lost Wages, heading east for Pecos Gulch Conference Center. With the wind blowing west to east, he averages a land speed of 30 km/hr. This will allow him to arrive in 4 hours, just as the conference begins. Meanwhile, Rhaina is 160 km north of Lost Wages. At the moment of Paul's lift off, Rhaina hops into an off-roading vehicle and heads directly for the conference center. At what average speed must she travel to arrive at the same time Paul does? [h](#)

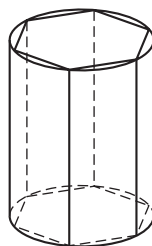
### Career

### CONNECTION

Meteorologists study the weather and the atmosphere. They also look at air quality, oceanic influence on weather, changes in climate over time, and even other planetary climates. They make forecasts using satellite photographs, weather balloons, contour maps, and mathematics to calculate wind speed or the arrival of a storm.



4. A 25-foot ladder is placed against a building. The bottom of the ladder is 7 feet from the building. If the top of the ladder slips down 4 feet, how many feet will the bottom slide out? (It is not 4 feet.) [h](#)
5. The front and back walls of an A-frame cabin are isosceles triangles, each with a base measuring 10 m and legs measuring 13 m. The entire front wall is made of glass 1 cm thick that cost \$120/m<sup>2</sup>. What did the glass for the front wall cost? [h](#)
6. A regular hexagonal prism fits perfectly inside a cylindrical box with diameter 6 cm and height 10 cm. What is the surface area of the prism? What is the surface area of the cylinder? [h](#)



7. Find the perimeter of an equilateral triangle whose median measures 6 cm.
8. **APPLICATION** According to the Americans with Disabilities Act, the slope of a wheelchair ramp must be no greater than  $\frac{1}{12}$ . What is the length of ramp needed to gain a height of 4 feet? Read the Science Connection on the top of page 484 and then figure out how much force is required to go up the ramp if a person and a wheelchair together weigh 200 pounds.



## Science

### CONNECTION

It takes less effort to roll objects up an *inclined plane*, or ramp, than to lift them straight up. *Work* is a measure of force applied over distance. You can calculate it as a product of force and distance. For example, a force of 100 pounds is required to hold up a 100-pound object. The work to lift it 2 feet is 200 foot-pounds. But if you use a 4-foot-long ramp to roll it up, you'll do the 200 foot-pounds of work over a 4-foot distance. So you need to apply only 50 pounds of force at any given moment.



For Exercises 9 and 10, refer to the above Science Connection about inclined planes.

9. Compare what it would take to lift an object these three different ways.
  - a. How much work, in foot-pounds, is necessary to lift 80 pounds straight up 2 feet?
  - b. If a ramp 4 feet long is used to raise the 80 pounds up 2 feet, how much force, in pounds, will it take?
  - c. If a ramp 8 feet long is used to raise the 80 pounds up 2 feet, how much force, in pounds, will it take?
10. If you can exert only 70 pounds of force and you need to lift a 160-pound steel drum up 2 feet, what is the minimum length of ramp you should set up?

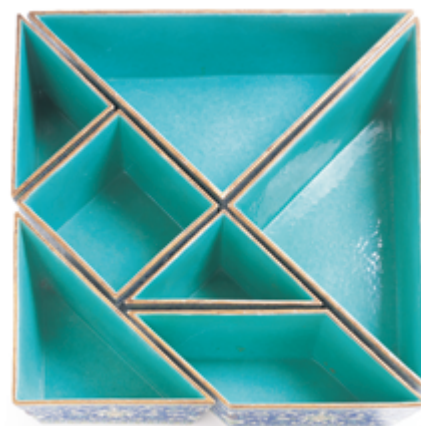
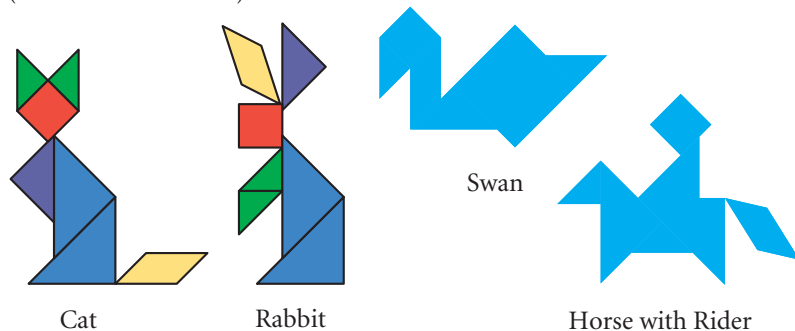
## Review

### Recreation

### CONNECTION

This set of enameled porcelain *qi qiao* bowls can be arranged to form a 37-by-37 cm square (as shown) or other shapes, or used separately. Each bowl is 10 cm deep. Dishes of this type are usually used to serve candies, nuts, dried fruits, and other snacks on special occasions.

The *qi qiao*, or tangram puzzle, originated in China and consists of seven pieces—five isosceles right triangles, a square, and a parallelogram. The puzzle involves rearranging the pieces into a square, or hundreds of other shapes (a few are shown below).



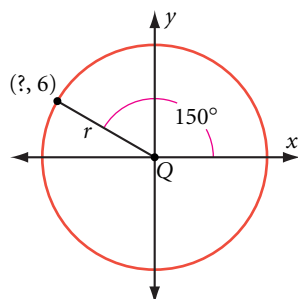
Private collection, Berkeley, California.  
Photo by Cheryl Fenton.

11. If the area of the red square piece is  $4 \text{ cm}^2$ , what are the dimensions of the other six pieces?

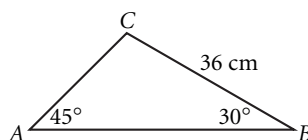


12. Make a set of your own seven tangram pieces and create the Cat, Rabbit, Swan, and Horse with Rider as shown on page 484.

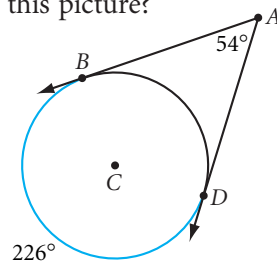
13. Find the radius of circle Q.



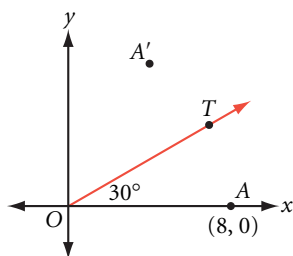
14. Find the length of  $\overline{AC}$ .



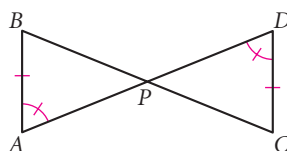
15. The two rays are tangent to the circle. What's wrong with this picture?



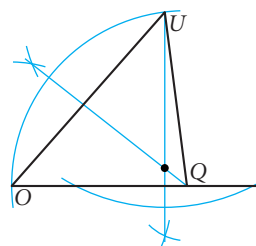
16. In the figure below, point  $A'$  is the image of point A after a reflection over  $\overleftrightarrow{OT}$ . What are the coordinates of  $A'$ ?



17. Which congruence shortcut can you use to show that  $\triangle ABP \cong \triangle DCP$ ?



18. Identify the point of concurrency in  $\triangle QUO$  from the construction marks.



19. In parallelogram QUID,  $m\angle Q = 2x + 5^\circ$  and  $m\angle I = 4x - 55^\circ$ . What is  $m\angle U$ ?

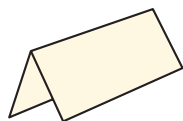
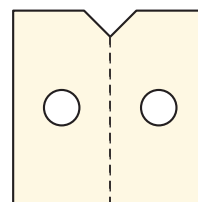
20. In  $\triangle PRO$ ,  $m\angle P = 70^\circ$  and  $m\angle R = 45^\circ$ . Which side of the triangle is the shortest?

## IMPROVING YOUR VISUAL THINKING SKILLS

### Fold, Punch, and Snip

A square sheet of paper is folded vertically, a hole is punched out of the center, and then one of the corners is snipped off. When the paper is unfolded it will look like the figure at right.

Sketch what a square sheet of paper will look like when it is unfolded after the following sequence of folds, punches, and snips.



Fold once.



Fold twice.



Snip double-fold corner.



Punch opposite corner.



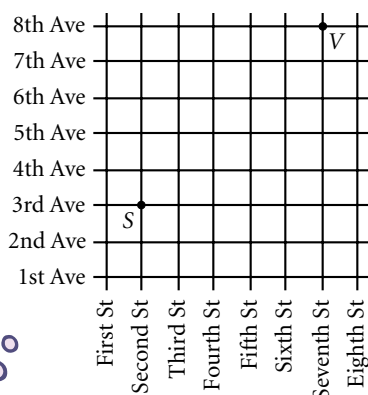


# Distance in Coordinate Geometry

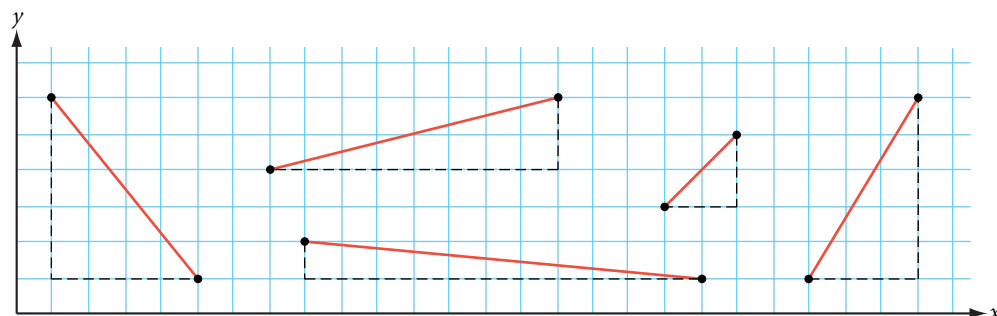
*We talk too much; we should talk less and draw more.*

JOHANN WOLFGANG  
VON GOETHE

Viki is standing on the corner of Seventh Street and 8th Avenue, and her brother Scott is on the corner of Second Street and 3rd Avenue. To find her shortest sidewalk route to Scott, Viki can simply count blocks. But if Viki wants to know her diagonal distance to Scott, she would need the Pythagorean Theorem to measure across blocks.



You can think of a coordinate plane as a grid of streets with two sets of parallel lines running perpendicular to each other. Every segment in the plane that is not in the  $x$ - or  $y$ -direction is the hypotenuse of a right triangle whose legs are in the  $x$ - and  $y$ -directions. So you can use the Pythagorean Theorem to find the distance between any two points on a coordinate plane.



## Investigation 1 The Distance Formula

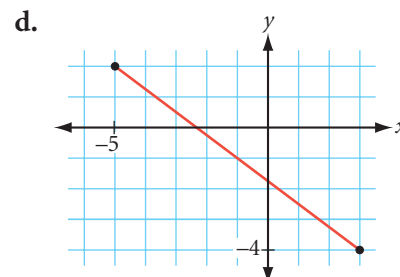
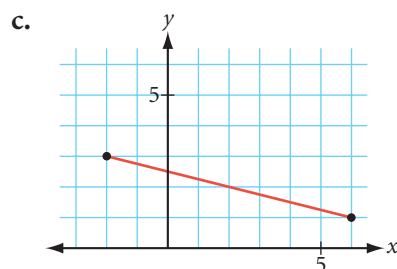
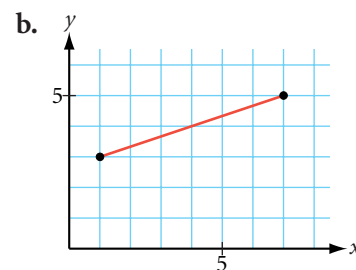
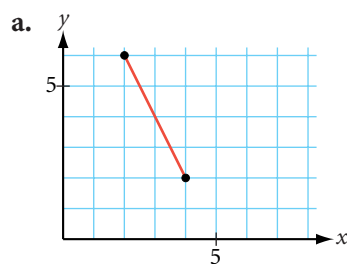
### You will need

- graph paper

#### Step 1

Copy graphs a–d from the next page onto your own graph paper. Use each segment as the hypotenuse of a right triangle. Draw the legs along the grid lines. Find the length of each segment.





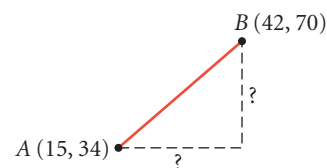
Step 2

Graph each pair of points, then find the distances between them.

a.  $(-1, -2), (11, -7)$

b.  $(-9, -6), (3, 10)$

What if the points are so far apart that it's not practical to plot them? For example, what is the distance between the points  $A(15, 34)$  and  $B(42, 70)$ ? A formula that uses the coordinates of the given points would be helpful. To find this formula, you first need to find the lengths of the legs in terms of the  $x$ - and  $y$ -coordinates. From your work with slope triangles, you know how to calculate horizontal and vertical distances.



Step 3

Write an expression for the length of the horizontal leg using the  $x$ -coordinates.

Step 4

Write a similar expression for the length of the vertical leg using the  $y$ -coordinates.

Step 5

Use your expressions from Steps 3 and 4, and the Pythagorean Theorem, to find the distance between points  $A(15, 34)$  and  $B(42, 70)$ .

Step 6

Generalize what you have learned about the distance between two points in a coordinate plane. Copy and complete the conjecture below.

### Distance Formula

C-86

The distance between points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is given by

$$(AB)^2 = (\underline{\quad})^2 + (\underline{\quad})^2 \text{ or } AB = \sqrt{(\underline{\quad})^2 + (\underline{\quad})^2}$$

Let's look at an example to see how you can apply the distance formula.

**EXAMPLE A** Find the distance between  $A(8, 15)$  and  $B(-7, 23)$ .

► **Solution**

$$\begin{aligned}(AB)^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 && \text{The distance formula.} \\ &= (-7 - 8)^2 + (23 - 15)^2 && \text{Substitute 8 for } x_1, 15 \text{ for } y_1, -7 \text{ for } x_2, \text{ and } 23 \text{ for } y_2. \\ &= (-15)^2 + (8)^2 && \text{Subtract.} \\ (AB)^2 &= 289 && \text{Square } -15 \text{ and } 8 \text{ and add.} \\ AB &= 17 && \text{Take the square root of both sides.}\end{aligned}$$

The distance formula is also used to write the equation of a circle.

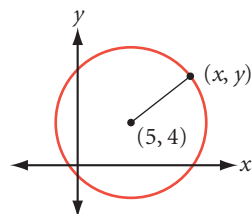
**EXAMPLE B** Write an equation for the circle with center  $(5, 4)$  and radius 7 units.

► **Solution**

Let  $(x, y)$  represent any point on the circle. The distance from  $(x, y)$  to the circle's center,  $(5, 4)$ , is 7. Substitute this information in the distance formula.

$$(x - 5)^2 + (y - 4)^2 = 7^2$$

Substitute  $(x, y)$  for  $(x_2, y_2)$ .      Substitute  $(5, 4)$  for  $(x_1, y_1)$ .      Substitute 7 as the distance.



So, the equation in standard form is  $(x - 5)^2 + (y - 4)^2 = 7^2$ .



## Investigation 2

### The Equation of a Circle

#### You will need

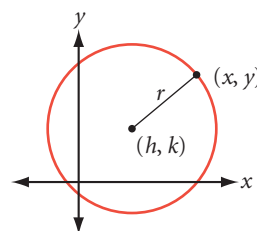
- graph paper

- Step 1 Given its center and radius, graph each circle on graph paper.
- a. Center =  $(1, -2)$ ,  $r = 8$       b. Center =  $(0, 2)$ ,  $r = 6$
- c. Center =  $(-3, -4)$ ,  $r = 10$
- Step 2 Select any point on each circle; label it  $(x, y)$ . Use the distance formula to write an equation expressing the distance between the center of each circle and  $(x, y)$ .
- Step 3 Copy and complete the conjecture for the equation of a circle.

#### Equation of a Circle

The equation of a circle with radius  $r$  and center  $(h, k)$  is

$$(x - \underline{\quad})^2 + (y - \underline{\quad})^2 = (\underline{\quad})^2$$



C-87

Let's look at an example that uses the equation of a circle in reverse.

**EXAMPLE C** Find the center and radius of the circle  $(x + 2)^2 + (y - 5)^2 = 36$ .

► **Solution**

Rewrite the equation of the circle in the standard form.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - (-2))^2 + (y - 5)^2 = 6^2$$

Identify the values of  $h$ ,  $k$ , and  $r$ . The center is  $(-2, 5)$  and the radius is 6.

## EXERCISES

In Exercises 1–3, find the distance between each pair of points.

1.  $(10, 20), (13, 16)$

2.  $(15, 37), (42, 73)$

3.  $(-19, -16), (-3, 14)$

4. Look back at the diagram of Viki's and Scott's locations on page 486. Assume each block is approximately 50 meters long. What is the shortest distance from Viki to Scott to the nearest meter?

5. Find the perimeter of  $\triangle ABC$  with vertices  $A(2, 4)$ ,  $B(8, 12)$ , and  $C(24, 0)$ .

6. Determine whether  $\triangle DEF$  with vertices  $D(6, -6)$ ,  $E(39, -12)$ , and  $F(24, 18)$  is scalene, isosceles, or equilateral.

For Exercises 7 and 8, find the equation of the circle.

7. Center =  $(0, 0)$ ,  $r = 4$

8. Center =  $(2, 0)$ ,  $r = 5$

For Exercises 9 and 10, find the radius and center of the circle.

9.  $(x - 2)^2 + (y + 5)^2 = 6^2$

10.  $x^2 + (y - 1)^2 = 81$

11. The center of a circle is  $(3, -1)$ . One point on the circle is  $(6, 2)$ . Find the equation of the circle. (h)

12. **Mini-Investigation** How would you find the distance between two points in a three-dimensional coordinate system? Investigate and make a conjecture. (h)

a. What is the distance from the origin  $(0, 0, 0)$  to  $(2, -1, 3)$ ?

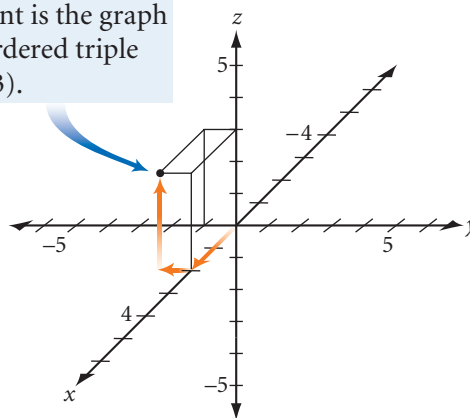
b. What is the distance between  $P(1, 2, 3)$  and  $Q(5, 6, 15)$ ?

c. Complete this conjecture:

If  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$  are two points in a three-dimensional coordinate system, then the distance  $AB$  is

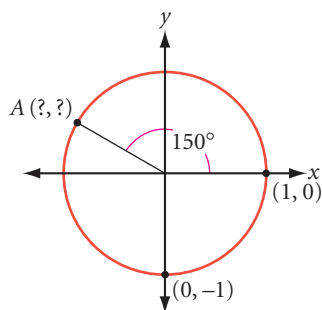
$$\sqrt{(\quad)^2 + (\quad)^2 + (\quad)^2}.$$

This point is the graph of the ordered triple  $(2, -1, 3)$ .

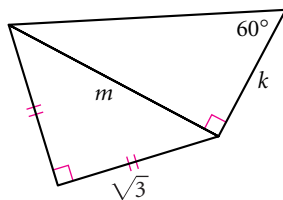


## Review

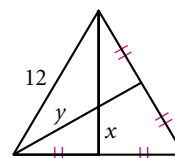
13. Find the coordinates of  $A$ .



14.  $k = ?$ ,  $m = ?$  (h)



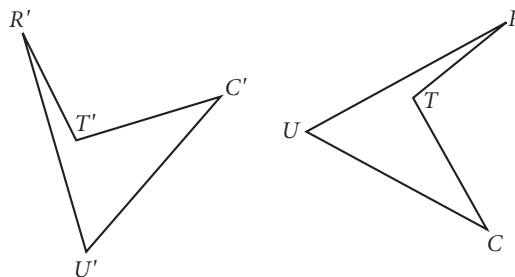
15. The large triangle is equilateral. Find  $x$  and  $y$ .



16. Antonio is a biologist studying life in a pond. He needs to know how deep the water is. He notices a water lily sticking straight up from the water, whose blossom is 8 cm above the water's surface. Antonio pulls the lily to one side, keeping the stem straight, until the blossom touches the water at a spot 40 cm from where the stem first broke the water's surface. How is Antonio able to calculate the depth of the water? What is the depth? (h)



17.  $C'U'R'T'$  is the image of  $CURT$  under a rotation transformation. Copy the polygon and its image onto patty paper. Find the center of rotation and the measure of the angle of rotation. Explain your method.

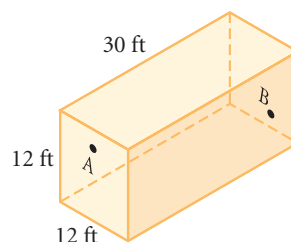


## IMPROVING YOUR VISUAL THINKING SKILLS

### The Spider and the Fly

(attributed to the British puzzlist Henry E. Dudeney, 1857–1930)

In a rectangular room, measuring 30 by 12 by 12 feet, a spider is at point  $A$  on the middle of one of the end walls, 1 foot from the ceiling. A fly is at point  $B$  on the center of the opposite wall, 1 foot from the floor. What is the shortest distance that the spider must crawl to reach the fly, which remains stationary? The spider never drops or uses its web, but crawls fairly.



# Exploration

## Ladder Climb

Suppose a house painter rests a 20-foot ladder against a building, then decides the ladder needs to rest 1 foot higher against the building. Will moving the ladder 1 foot toward the building do the job? If it needs to be 2 feet lower, will moving the ladder 2 feet away from the building do the trick? Let's investigate.



### Activity

#### Climbing the Wall

##### You will need

- a graphing calculator

Sketch a ladder leaning against a vertical wall, with the foot of the ladder resting on horizontal ground. Label the sketch using  $y$  for height reached by the ladder and  $x$  for the distance from the base of the wall to the foot of the ladder.

- |        |  |
|--------|--|
| Step 1 | Write an equation relating $x$ , $y$ , and the length of the ladder and solve it for $y$ . You now have a function for the height reached by the ladder in terms of the distance from the wall to the foot of the ladder. Enter this equation into your calculator.  |
| Step 2 | Before you graph the equation, think about the settings you'll want for the graph window. What are the greatest and least values possible for $x$ and $y$ ? Enter reasonable settings, then graph the equation.  |
| Step 3 | Describe the shape of the graph.   |
| Step 4 | Trace along the graph, starting at $x = 0$ . Record values (rounded to the nearest 0.1 unit) for the height reached by the ladder when $x = 3, 6, 9$ , and $12$ . If you move the foot of the ladder away from the wall 3 feet at a time, will each move result in the same change in the height reached by the ladder? Explain. |
| Step 5 | Find the value for $x$ that gives a $y$ -value approximately equal to $x$ . How is this value related to the length of the ladder? Sketch the ladder in this position. What angle does the ladder make with the ground?  |
| Step 6 | Should you lean a ladder against a wall in such a way that $x$ is greater than $y$ ? Explain. How does your graph support your explanation?  |

*You must do things you think  
you cannot do.*

ELEANOR ROOSEVELT

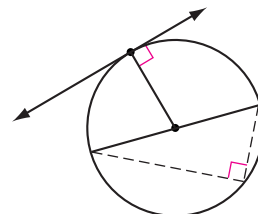
# Circles and the Pythagorean Theorem

In Chapter 6, you discovered a number of properties that involved right angles in and around circles. In this lesson you will use the conjectures you made, along with the Pythagorean Theorem, to solve some challenging problems. Let's review two of the most useful conjectures.

**Tangent Conjecture:** A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

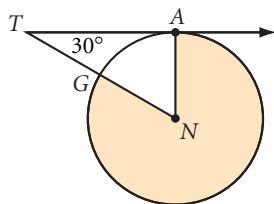
**Angles Inscribed in a Semicircle Conjecture:** Angles inscribed in a semicircle are right angles.

Here are two examples that use these conjectures along with the Pythagorean Theorem.



## EXAMPLE A

$\overline{TA}$  is tangent to circle  $N$  at  $A$ .  $TA = 12\sqrt{3}$  cm. Find the area of the shaded region.

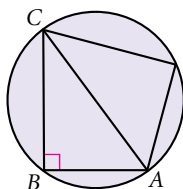


### ► Solution

$\overline{TA}$  is tangent at  $A$ , so  $\angle TAN$  is a right angle and  $\triangle TAN$  is a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle. The longer leg is  $12\sqrt{3}$  cm, so the shorter leg (also the radius of the circle) is 12 cm. The area of the entire circle is  $144\pi$  cm<sup>2</sup>. The area of the shaded region is  $\frac{360 - 60}{360}$ , or  $\frac{5}{6}$ , of the area of the circle. Therefore the shaded area is  $\frac{5}{6}(144\pi)$ , or  $120\pi$  cm<sup>2</sup>.

## EXAMPLE B

$AB = 6$  cm and  $BC = 8$  cm. Find the area of the circle.



### ► Solution

Inscribed angle  $ABC$  is a right angle, so  $\overline{ABC}$  is a semicircle and  $\overline{AC}$  is a diameter. By the Pythagorean Theorem, if  $AB = 6$  cm and  $BC = 8$  cm, then  $AC = 10$  cm. Therefore the radius of the circle is 5 cm and the area of the circle is  $25\pi$  cm<sup>2</sup>.



# EXERCISES

You will need



**Construction tools**  
for Exercise 20

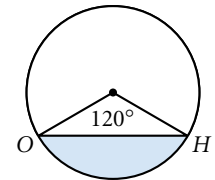
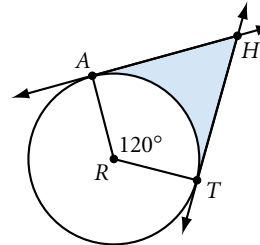
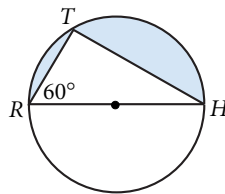
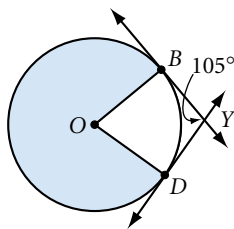
In Exercises 1–4, find the area of the shaded region in each figure. Assume lines that appear tangent are tangent at the labeled points.

1.  $OD = 24$  cm h

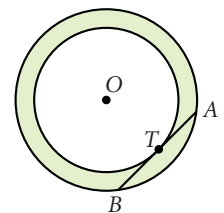
2.  $HT = 8\sqrt{3}$  cm

3.  $HA = 8\sqrt{3}$  cm h

4.  $HO = 8\sqrt{3}$  cm



5. A 3-meter-wide circular track is shown at right. The radius of the inner circle is 12 meters. What is the longest straight path that stays on the track? (In other words, find  $AB$ .) h

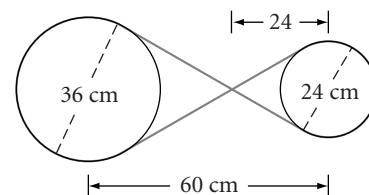


6. An annulus has a 36 cm chord of the outer circle that is also tangent to the inner concentric circle. Find the area of the annulus.

7. In her latest expedition, Ertha Diggs has uncovered a portion of circular, terra-cotta pipe that she believes is part of an early water drainage system. To find the diameter of the original pipe, she lays a meterstick across the portion and measures the length of the chord at 48 cm. The depth of the portion from the midpoint of the chord is 6 cm. What was the pipe's original diameter?



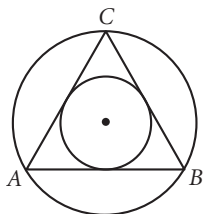
8. **APPLICATION** A machinery belt needs to be replaced. The belt runs around two wheels, crossing between them so that the larger wheel turns the smaller wheel in the opposite direction. The diameter of the larger wheel is 36 cm, and the diameter of the smaller is 24 cm. The distance between the centers of the two wheels is 60 cm. The belt crosses 24 cm from the center of the smaller wheel. What is the length of the belt? h



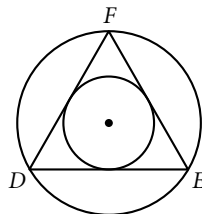
9. A circle of radius 6 has chord  $\overline{AB}$  of length 6. If point  $C$  is selected randomly on the circle, what is the probability that  $\triangle ABC$  is obtuse?

In Exercises 10 and 11, each triangle is equilateral. Find the area of the inscribed circle and the area of the circumscribed circle. How many times greater is the area of the circumscribed circle than the area of the inscribed circle?

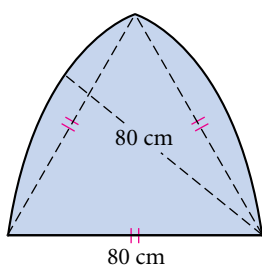
10.  $AB = 6$  cm h



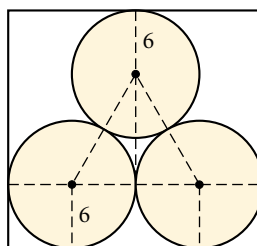
11.  $DE = 2\sqrt{3}$  cm



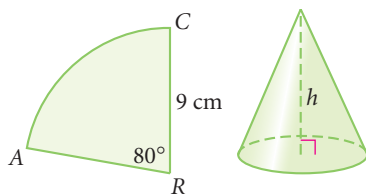
12. The Gothic arch is based on the equilateral triangle. If the base of the arch measures 80 cm, what is the area of the shaded region?



13. Each of three circles of radius 6 cm is tangent to the other two, and they are inscribed in a rectangle, as shown. What is the height of the rectangle?



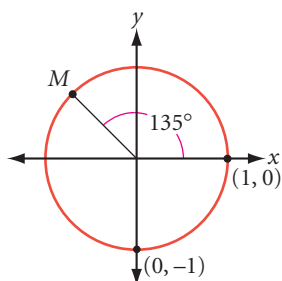
14. Sector  $ARC$  has a radius of 9 cm and an angle that measures  $80^\circ$ . When sector  $ARC$  is cut out and  $\overline{AR}$  and  $\overline{RC}$  are taped together, they form a cone. The length of  $\widehat{AC}$  becomes the circumference of the base of the cone. What is the height of the cone? h



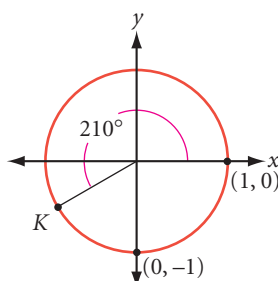
15. **APPLICATION** Will plans to use a circular cross section of wood to make a square table. The cross section has a circumference of 336 cm. To the nearest centimeter, what is the side length of the largest square that he can cut from it?




16. Find the coordinates of point  $M$ .



17. Find the coordinates of point  $K$ .



## Review

18. Find the equation of a circle with center  $(3, 3)$  and radius 6.
19. Find the radius and center of a circle with the equation  $x^2 + y^2 - 2x + 1 = 100$ . 
20. **Construction** Construct a circle and a chord in a circle. With compass and straightedge, construct a second chord parallel and congruent to the first chord. Explain your method.
21. Explain why the opposite sides of a regular hexagon are parallel.
22. Find the rule for this number pattern:
- $$\begin{array}{l} 1 \cdot 3 - 3 = 4 \cdot 0 \\ 2 \cdot 4 - 3 = 5 \cdot 1 \\ 3 \cdot 5 - 3 = 6 \cdot 2 \\ 4 \cdot 6 - 3 = 7 \cdot 3 \\ 5 \cdot 7 - 3 = 8 \cdot 4 \\ \vdots \\ n \cdot (\underline{\quad}) - (\underline{\quad}) = (\underline{\quad}) \cdot (\underline{\quad}) \end{array}$$
23. **APPLICATION** Felice wants to determine the diameter of a large heating duct. She places a carpenter's square up to the surface of the cylinder, and the length of each tangent segment is 10 inches.
- What is the diameter? Explain your reasoning.
  - Describe another way she can find the diameter of the duct.



## IMPROVING YOUR REASONING SKILLS

### Reasonable 'rithmetic I

Each letter in these problems represents a different digit.

1. What is the value of  $B$ ?

$$\begin{array}{r} 3 \quad 7 \quad 2 \\ 3 \quad 8 \quad 4 \\ + \quad 9 \quad B \quad 4 \\ \hline C \quad 7 \quad C \quad A \end{array}$$

2. What is the value of  $J$ ?

$$\begin{array}{r} E \quad F \quad 6 \\ \times \quad D \quad 7 \\ \hline D \quad D \quad F \quad D \\ J \quad E \quad D \\ \hline H \quad G \quad E \quad D \end{array}$$



If 50 years from now you've forgotten everything else you learned in geometry, you'll probably still remember the Pythagorean Theorem. (Though let's hope you don't really forget everything else!) That's because it has practical applications in the mathematics and science that you encounter throughout your education.

It's one thing to remember the equation  $a^2 + b^2 = c^2$ . It's another to know what it means and to be able to apply it. Review your work from this chapter to be sure you understand how to use special triangle shortcuts and how to find the distance between two points in a coordinate plane.



## EXERCISES

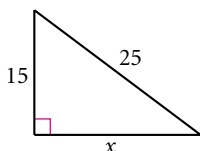
You will need



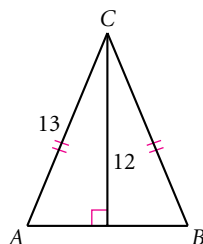
Construction tools  
for Exercise 30

For Exercises 1–4, measurements are given in centimeters.

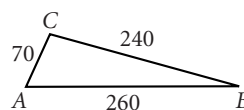
1.  $x = ?$



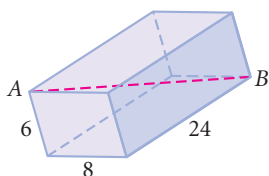
2.  $AB = ?$



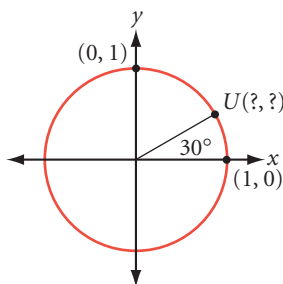
3. Is  $\triangle ABC$  an acute, obtuse, or right triangle?



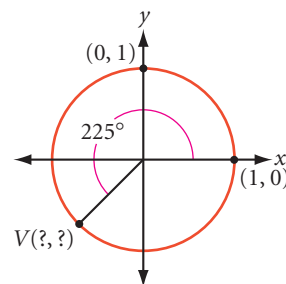
4. The solid is a rectangular prism.  $AB = ?$



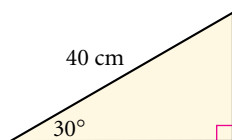
5. Find the coordinates of point U.



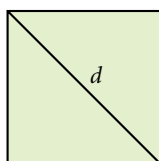
6. Find the coordinates of point V.



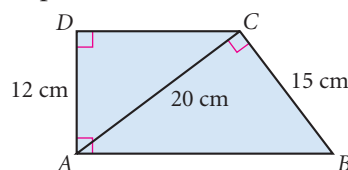
7. What is the area of the triangle?



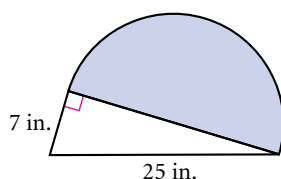
8. The area of this square is  $144 \text{ cm}^2$ . Find  $d$ .



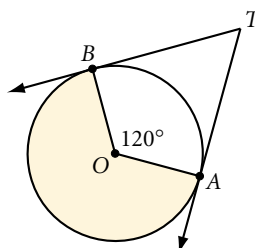
9. What is the area of trapezoid ABCD?



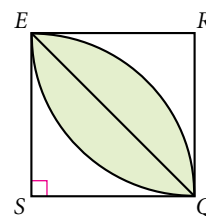
10. The arc is a semicircle. What is the area of the shaded region? [h](#)



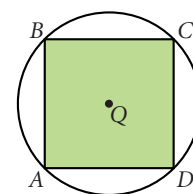
11. Rays  $TA$  and  $TB$  are tangent to circle  $O$  at  $A$  and  $B$  respectively, and  $BT = 6\sqrt{3}$  cm. What is the area of the shaded region?



12. The quadrilateral is a square, and  $QE = 2\sqrt{2}$  cm. What is the area of the shaded region? [h](#)



13. The area of circle  $Q$  is  $350 \text{ cm}^2$ . Find the area of square  $ABCD$  to the nearest  $0.1 \text{ cm}^2$ .



14. Determine whether  $\triangle ABC$  with vertices  $A(3, 5)$ ,  $B(11, 3)$ , and  $C(8, 8)$  is an equilateral, isosceles, or isosceles right triangle.

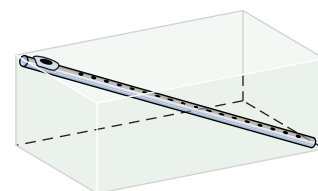
15. Sagebrush Sally leaves camp on her dirt bike traveling east at  $60 \text{ km/hr}$  with a full tank of gas. After 2 hours, she stops and does a little prospecting—with no luck. So she heads north for 2 hours at  $45 \text{ km/hr}$ . She stops again, and this time hits pay dirt. Sally knows that she can travel at most  $350 \text{ km}$  on one tank of gas. Does she have enough fuel to get back to camp? If not, how close can she get? [h](#)



16. A parallelogram has sides measuring  $8.5 \text{ cm}$  and  $12 \text{ cm}$ , and a diagonal measuring  $15 \text{ cm}$ . Is the parallelogram a rectangle? If not, is the  $15 \text{ cm}$  diagonal the longer or shorter diagonal? [h](#)

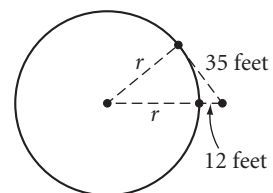
17. After an argument, Peter and Paul walk away from each other on separate paths at a right angle to each other. Peter is walking  $2 \text{ km/hr}$ , and Paul is walking  $3 \text{ km/hr}$ . After 20 min, Paul sits down to think. After 30 min, Peter stops. Both decide to apologize. How far apart are they? How long will it take them to reach each other if they both start running straight toward each other at  $5 \text{ km/hr}$ ?

18. Flora is away at camp and wants to mail her flute back home. The flute is  $24$  inches long. Will it fit diagonally within a box whose inside dimensions are  $12$  by  $16$  by  $14$  inches?



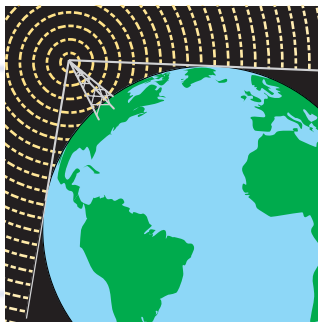
19. To the nearest foot, find the original height of a fallen flagpole that cracked and fell as if hinged, forming an angle of  $45$  degrees with the ground. The tip of the pole hit the ground  $12$  feet from its base.

20. You are standing 12 feet from a cylindrical corn-syrup storage tank. The distance from you to a point of tangency on the tank is 35 feet. What is the radius of the tank?

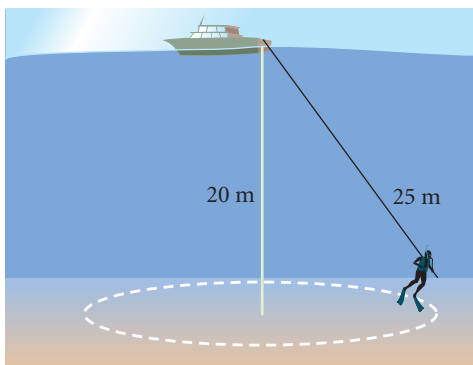


### Technology CONNECTION

Radio and TV stations broadcast from high towers. Their signals are picked up by radios and TVs in homes within a certain radius. Because Earth is spherical, these signals don't get picked up beyond the point of tangency.

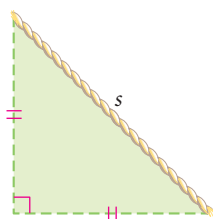


21. **APPLICATION** Read the Technology Connection above. What is the maximum broadcasting radius from a radio tower 1800 feet tall (approximately 0.34 mile)? The radius of Earth is approximately 3960 miles, and you can assume the ground around the tower is nearly flat. Round your answer to the nearest 10 miles.
22. A diver hooked to a 25-meter line is searching for the remains of a Spanish galleon in the Caribbean Sea. The sea is 20 meters deep and the bottom is flat. What is the area of circular region that the diver can explore?
23. What are the lengths of the two legs of a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle if the length of the hypotenuse is  $12\sqrt{3}$ ?
24. Find the side length of an equilateral triangle with an area of  $36\sqrt{3} \text{ m}^2$ .
25. Find the perimeter of an equilateral triangle with a height of  $7\sqrt{3}$ .
26. Al baked brownies for himself and his two sisters. He divided the square pan of brownies into three parts. He measured three  $30^\circ$  angles at one of the corners so that two pieces formed right triangles and the middle piece formed a kite. Did he divide the pan of brownies equally? Draw a sketch and explain your reasoning.
27. A circle has a central angle  $AOB$  that measures  $80^\circ$ . If point  $C$  is selected randomly on the circle, what is the probability that  $\triangle ABC$  is obtuse?

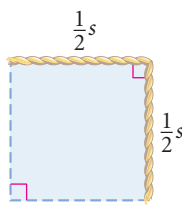




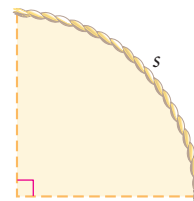
28. One of the sketches below shows the greatest area that you can enclose in a right-angled corner with a rope of length  $s$ . Which one? Explain your reasoning.



A triangle

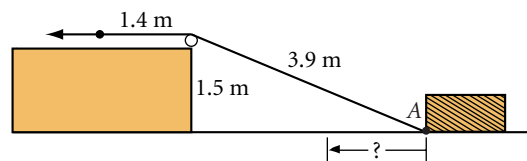


A square



A quarter-circle

29. A wire is attached to a block of wood at point A. The wire is pulled over a pulley as shown. How far will the block move if the wire is pulled 1.4 meters in the direction of the arrow?



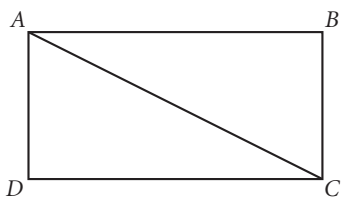
## MIXED REVIEW

30. **Construction** Construct an isosceles triangle that has a base length equal to half the length of one leg.
31. In a regular octagon inscribed in a circle, how many diagonals pass through the center of the circle? In a regular nonagon? a regular 20-gon? What is the general rule?
32. A bug clings to a point two inches from the center of a spinning fan blade. The blade spins around once per second. How fast does the bug travel in inches per second?
- In Exercises 33–40, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.
33. The area of a rectangle and the area of a parallelogram are both given by the formula  $A = bh$ , where  $A$  is the area,  $b$  is the length of the base, and  $h$  is the height.
34. When a figure is reflected over a line, the line of reflection is perpendicular to every segment joining a point on the original figure with its image.
35. In an isosceles right triangle, if the legs have length  $x$ , then the hypotenuse has length  $x\sqrt{3}$ .
36. The area of a kite or a rhombus can be found by using the formula  $A = (0.5)d_1d_2$ , where  $A$  is the area and  $d_1$  and  $d_2$  are the lengths of the diagonals.
37. If the coordinates of points  $A$  and  $B$  are  $(x_1, y_1)$  and  $(x_2, y_2)$ , respectively, then  $AB = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ .
38. A glide reflection is a combination of a translation and a rotation.

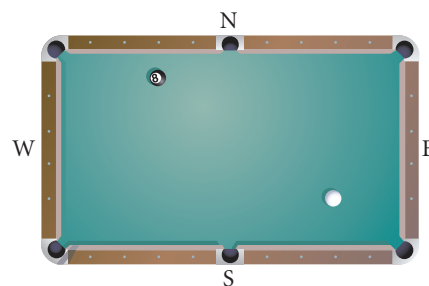
- 39.** Equilateral triangles, squares, and regular octagons can be used to create monohedral tessellations.
- 40.** In a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle, if the shorter leg has length  $x$ , then the longer leg has length  $x\sqrt{3}$  and the hypotenuse has length  $2x$ .

In Exercises 41–46, select the correct answer.

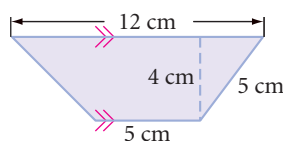
- 41.** The hypotenuse of a right triangle is always ?.
- A. opposite the smallest angle and is the shortest side.
  - B. opposite the largest angle and is the shortest side.
  - C. opposite the smallest angle and is the longest side.
  - D. opposite the largest angle and is the longest side.
- 42.** The area of a triangle is given by the formula ?, where  $A$  is the area,  $b$  is the length of the base, and  $h$  is the height.
- A.  $A = bh$
  - B.  $A = \frac{1}{2}bh$
  - C.  $A = 2bh$
  - D.  $A = b^2h$
- 43.** If the lengths of the three sides of a triangle satisfy the Pythagorean equation, then the triangle must be a(n) ? triangle.
- A. right
  - B. acute
  - C. obtuse
  - D. scalene
- 44.** The ordered pair rule  $(x, y) \rightarrow (y, x)$  is a ?.
- A. reflection over the  $x$ -axis
  - B. reflection over the  $y$ -axis
  - C. reflection over the line  $y = x$
  - D. rotation  $90^\circ$  about the origin
- 45.** The composition of two reflections over two intersecting lines is equivalent to ?.
- A. a single reflection
  - B. a translation
  - C. a rotation
  - D. no transformation
- 46.** The total surface area of a cone is equal to ?, where  $r$  is the radius of the circular base and  $l$  is the slant height.
- A.  $\pi r^2 + 2\pi r$
  - B.  $\pi rl$
  - C.  $\pi rl + 2\pi r$
  - D.  $\pi rl + \pi r^2$
- 47.** Create a flowchart proof to show that the diagonal of a rectangle divides the rectangle into two congruent triangles.



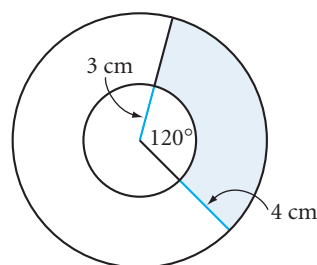
- 48.** Copy the ball positions onto patty paper.
- At what point on the S cushion should a player aim so that the cue ball bounces off and strikes the 8-ball? Mark the point with the letter *A*.
  - At what point on the W cushion should a player aim so that the cue ball bounces off and strikes the 8-ball? Mark the point with the letter *B*.



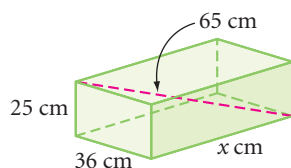
- 49.** Find the area and the perimeter of the trapezoid.



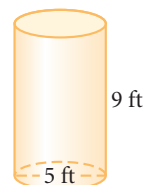
- 50.** Find the area of the shaded region.



- 51.** An Olympic swimming pool has length 50 meters and width 25 meters. What is the diagonal distance across the pool?
- 53.** The box below has dimensions 25 cm, 36 cm, and  $x$  cm. The diagonal shown has length 65 cm. Find the value of  $x$ .

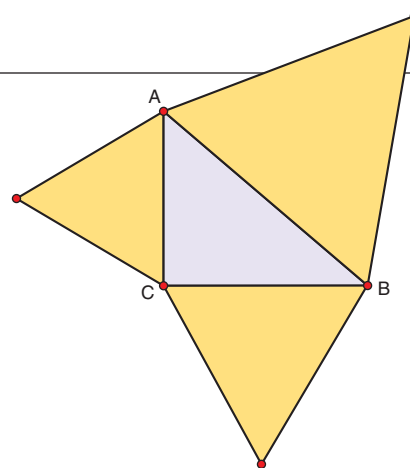


- 52.** The side length of a regular pentagon is 6 cm, and the apothem measures about 4.1 cm. What is the area of the pentagon?
- 54.** The cylindrical container below has an open top. Find the surface area of the container (inside and out) to the nearest square foot.

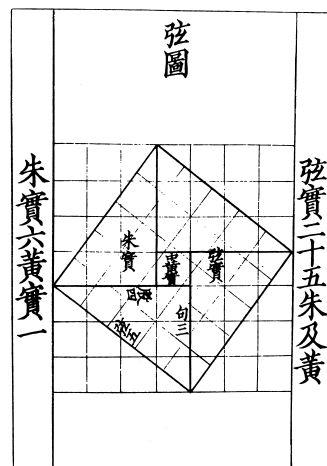


## TAKE ANOTHER LOOK

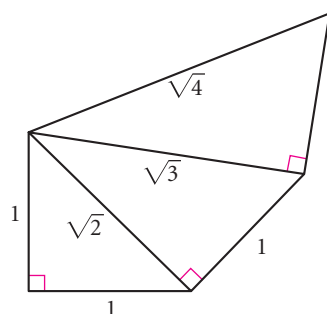
- Use geometry software to demonstrate the Pythagorean Theorem. Does your demonstration still work if you use a shape other than a square—for example, an equilateral triangle or a semicircle?
- Find Elisha Scott Loomis's *Pythagorean Proposition* and demonstrate one of the proofs of the Pythagorean Theorem from the book.



3. The *Zhoubi Suanjing*, one of the oldest sources of Chinese mathematics and astronomy, contains the diagram at right demonstrating the Pythagorean Theorem (called *gou gu* in China). Find out how the Chinese used and proved the *gou gu*, and present your findings.
4. Use the SSS Congruence Conjecture to verify the converse of the  $30^\circ$ - $60^\circ$ - $90^\circ$  Triangle Conjecture. That is, show that if a triangle has sides with lengths  $x$ ,  $x\sqrt{3}$ , and  $2x$ , then it is a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle.



5. Starting with an isosceles right triangle, use geometry software or a compass and straightedge to start a right triangle like the one shown. Continue constructing right triangles on the hypotenuse of the previous triangle at least five more times. Calculate the length of each hypotenuse and leave them in radical form.



## Assessing What You've Learned



**UPDATE YOUR PORTFOLIO** Choose a challenging project, Take Another Look activity, or exercise you did in this chapter and add it to your portfolio. Explain the strategies you used.



**ORGANIZE YOUR NOTEBOOK** Review your notebook and your conjecture list to be sure they are complete. Write a one-page chapter summary.



**WRITE IN YOUR JOURNAL** Why do you think the Pythagorean Theorem is considered one of the most important theorems in mathematics?



**WRITE TEST ITEMS** Work with group members to write test items for this chapter. Try to demonstrate more than one way to solve each problem.



**GIVE A PRESENTATION** Create a visual aid and give a presentation about the Pythagorean Theorem.